Rankings and values for team games

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Power index ranking

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Definition

Let $\phi, \psi : G \to \mathbb{R}^n$ be two indices. We say that ϕ and ψ **agree up to a constant** *if*

$$\phi_i(v) - \phi_j(v) = \psi_i(v) - \psi_j(v)$$

for every game $v \in G$ and players i and j.

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Let $\psi : G \to \mathbb{R}^n$ be any linear symmetric solution. Then there exists a unique solution $\phi : G \to \mathbb{R}^n$ which is linear, symmetric and efficient, agreeing up to a constant with ψ.

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Corollary

There exists a unique solution $\phi : G \to \mathbb{R}^n$ which is linear, symmetric, efficient and agrees up to a constant with the Banzhaf semi-value B. Moreover, it is given by,

$$\phi_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}} \sum_{S \ni i} (n-s)\left[v(S) - v(N \setminus S)\right]$$

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Corollary

Let

$$\psi_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S) \right]$$

be an arbitrary linear symmetric and null solution (see Dubey et al. (1981) or Hernández-Lamoneda et al. (2007)) then the unique linear symmetric efficient solution agrees up to a constant with ψ will be given by

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i} (n - s) \left[t_s v(S) - t_{n-s} v(N \setminus S) \right]$$

where

$$t_s = \frac{r_{s-1} + r_s}{n}.$$

Power index ranking

Definition

Let $\phi, \psi : G \to \mathbb{R}^n$ be two solutions (or indices). We say that $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ give the same power index ranking (PIR) if and only if for every game v, and all players i, j,

$$\phi_i(v) > \phi_j(v) \text{ iff } \psi_i(v) > \psi_j(v).$$

Power index ranking

Theorem

Let ϕ and ψ be two linear and symmetric indices. Then $\phi \overset{PIR}{=} \psi$ implies there exists $\lambda > 0$ such that $\phi \overset{uptok}{=} \lambda\psi$. The converse is also true.

In other words, two linear symmetric indices, ϕ and ψ, rank all players, for all games, in the same order, if and only if there exists a function $k : G \rightarrow \mathbb{R}^n$ and a positive constant λ, such that

$$\phi_i(v) = \lambda\psi_i(v) + k(v)$$

for every v and every i.

Team games

- For every $s : 1, \ldots, n$, let

$$G_s = \{v \in G \mid \; v(S) = 0 \text{ if } |S| \neq s\},$$

we think of the elements in G_s as "team games" with s players.

- **Basketball Example**
  - N set of players in the NBA.
  - $s = 5$.
  - $v(S)$ worth of the team S.
  - $I(v)$ ranking of the players.

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- **Double sculls example**
  - $N = \{1, \ldots, n\}$ set of rowers.
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PIR for team games

- Let $I : G_s \longrightarrow \mathbb{R}^n$ be given by

$$I_i(v) = \frac{1}{\binom{n-2}{s-1}} \sum_{S \ni i} v(S).$$

I is a linear, symmetric solution. Moreover, $\frac{1}{n-1} I$ agrees up to a constant with the restriction of Shapley's value to G_s.

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Theorem

Let $\phi : G_s \longrightarrow \mathbb{R}^n$ be any linear, symmetric index then ϕ has the same PIR as εI where $\varepsilon = -1, 0$ or 1.

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PIR for team games

The Poker example (cont.): Recall we have defined a game $v \in G_5^{52}$ that models the game of poker. Thus, the index $I : G_5^{52} \longrightarrow \mathbb{R}^{52}$ gives a ranking for the cards. Recall that all suits are assumed to have the same value, thus we may think that I takes values in $\mathbb{R}^{13}$. We list the rankings of the cards:

j	$I_j(v)$
A	0.236088
K	0.128311
Q	0.066628
J	0.029374
10	0.005330
9	-0.012679
8	-0.027257
7	-0.040026
6	-0.051830
5	-0.063794

Retiring players

- Let $v \in G_s^N$ be an arbitrary game, then if the player n drops out we get a new game $Rv \in G_s^{N\setminus\{n\}}$ by restriction: for $S \subset \{1, 2, \ldots, n-1\}$ let $Rv(S) := v(S)$.
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Theorem

Fix a player $k \in N$. Let $u \in \mathbb{R}^N$ and $x \in \mathbb{R}^{N\setminus\{k\}}$ be arbitrary vectors. If $2 \leq s \leq n-2$, then there exists a team game $v \in G_s^N$ such that:

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Retiring players

Rowing example revisited

Assume the following are the times (in minutes) recorded by each pair of oarsmen during the trials:

$v(\{1,5\}) =$	6.26
$v(\{2,5\}) =$	6.46
$v(\{3,5\}) =$	6.66
$v(\{4,5\}) =$	6.86
$v(\{1,2\}) =$	6.54
$v(\{1,3\}) =$	6.5
$v(\{1,4\}) =$	6.46
$v(\{2,3\}) =$	6.46
$v(\{2,4\}) =$	6.42
$v(\{3,4\}) =$	6.38

Retiring players

Then,

$$I(v) = \frac{1}{3}(25{,}76, 25{,}88, 26, 26{,}12, 26{,}24),$$

and so,

$$1 \prec 2 \prec 3 \prec 4 \prec 5$$

which means that the olympic team should consist of rowers $\{1, 2, 3\}$ in that order.

Unfortunately, a few days later it is discovered that rower # 5 failed the anti-doping test. Thus, all his trial times must be removed (even though he was the worse, by our index measurement, of all rowers) and the index for the restricted game (with the fifth rower "retired") is computed (the restricted game corresponds to the bottom 6 rows on the above table):

$$I(Rv) = \frac{1}{2}(19{,}5, 19{,}42, 19{,}34, 19{,}26).$$

Thus the new ranking is $1 \succ 2 \succ 3 \succ 4$. And, therefore, the revised olympic team should have rowers $\{4, 3, 2\}$ in that order.

Retiring players

- For a team game $v \in G_s$, define a team game $Fv \in G_{s-1}^{N\setminus\{n\}}$ by

$$Fv(T) = v(T \cup \{n\})$$

for every $T \subset N \setminus \{n\}$. Fv is a team game that has "freeçollaboration from n; $Rv \in G_s^{N\setminus\{n\}}$, as before, the restricted game. Then

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Theorem

$$(n-2)I^N(v) = (n-s-1)I^{N\setminus\{n\}}(Rv) + (s-1)I^{N\setminus\{n\}}(Fv).$$

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$G_s \oplus \mathbb{R}$ games

- We look at pairs $(v, c) \in G_s \oplus \mathbb{R}$ and solutions

$$\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$$

that are linear and symmetric. Here $1 \leq s < n$.

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Theorem

The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is 3-dimensional. Moreover, their general expression is given by

$$\phi_i(v, c) = Ac + B \sum_{S \ni i} v(S) - C \sum_{S \not\ni i} v(S)$$

for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$, for arbitrary $A, B, C \in \mathbb{R}$.

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$$\phi(v, c) \cdot 1_n = c.$$

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The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are also efficient is 1-dimensional. Their general expression is given by

$$\phi_i(v, c) = \frac{c}{n} + \lambda \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n - s} \right]$$

for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$, for arbitrary $\lambda \in \mathbb{R}$

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  for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$, for arbitrary $\lambda \in \mathbb{R}$

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- **Definition**

  (Efficiency Axiom) The solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is said to be efficient if

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  Given a vector $x \in \mathbb{R}^n$ and an integer $s : 1, \ldots, n$ we define a game, x^s, as follows:

$$x^s(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } |S| = s; \\ 0 & \text{if } |S| \neq s. \end{cases}$$

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Theorem

There exists a unique solution $\psi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ *which is linear, symmetric, efficient and natural. It is given by*

$$\psi_i(v,c) = \frac{c}{n} + \frac{n-1}{\binom{n}{s}} \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s} \right]$$

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$G_s \oplus \mathbb{R}$ games

Ejemplo del Europass.

- Se eligen s países europeos (donde $s = 3, 4$ o 5) de un conjunto N.
- $v(S)$ número de viajeros que selecciona a $S \subseteq N$, para viajar d días en los países de S.
- c es el monto total recolectado en la venta de Europass.
- La solución $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^N$ asigna a (v, c) un vector $\phi(v, c)$ donde $\phi_j(v, c)$ es el monto que le corresponde al país j.

Bankruptcy problem

- Recall that G_1 consists of games that can only take non zero values on cardinality one subsets of N. Thus, it is naturally identified with $\mathbb{R}^n$: $x(\{i\}) = x_i$.
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Definition

A bankruptcy game is an element $(x, c) \in G_1 \oplus \mathbb{R}$ such that

$$x(N) = \sum_{i=1}^{n} x_i \geq c.$$

- We interpret x_i as the amount that the ith creditor demands, whereas c is the total amount that may be repaid.
- All the results of the previous section apply to bankruptcy games. Let us summarize them together with an interpretation of their meaning.

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Bankruptcy problem

- There is a 3-dimensional space of linear, symmetric, solutions $\phi : G_1 \oplus \mathbb{R} \to \mathbb{R}^n$ for the bankruptcy problem. The general expression for such solutions is given by

$$\phi_i(x, c) = \alpha c + \beta x_i + \gamma x(N), \quad i : 1, \ldots, n;$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$.

- Efficiency (i.e., $\phi(x, c) \cdot 1_n = c$) means that the value of all the remaining goods, c, is divided among all creditors. There is a 1-dimensional space of linear, symmetric and efficient solutions given by

$$\phi(x, c) = \frac{c}{n} 1_n + \lambda(x - \frac{x(N)}{n} 1_n)$$

for arbitrary $\lambda \in \mathbb{R}$. Equivalently,

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- Naturalness (i.e., $\phi(x, x \cdot 1_n) = x$) is the axiom that says that if the total amount claimed by the creditors equals the value of the remaining goods, then each creditor receives what he claims. There is a 1-dimensional subspace of natural solutions (that are linear and symmetric) to the bankruptcy problem. They are given by

$$\phi(x, c) = x + \frac{(1-\alpha)}{n}(c - x \cdot 1_n)1_n$$

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Corollary

There exists a unique bankruptcy solution which is linear, symmetric, efficient and natural. It is given by

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References

- Dubey P., Neyman A. and Weber R.J. (1981). "Value Theory without Efficiency", *Mathematics of Operations Research*, Vol.6, No.1, pp. 122-128.
- Hernández-Lamoneda L., Juárez-García R. and Sánchez-Sánchez F. "Dissection of cooperative solutions in game theory using representation techniques", Int. J. Game Theory **35** (2007), pp. 395-426.
- Ruiz, L.M., Valenciano, F. and Zarzuelo J.M. (1996) "The least Square Prenucleoulus and the Least Square Nucleolus. Two values for TU Games Based on the Excess Vector", *International Journal of Game Theory*,Vol. 25, pp. 113-134.
- Saari D.G. and Sieberg K.K. (2001). "Some Surprising Properties of Power Indices", *Games and Economic Behavior*, vol. 36, 2, pp. 241-263.